January 22, 1880.

THE PRESIDENT in the Chair.

The Presents received were laid on the table, and thanks ordered for them.

The following Papers were read:—

I. "On certain Definite Integrals." No. 6. By W. H. L. Russell, F.R.S. Received November 15, 1879.

Continuing the methods of the last paper, we obtain the following results:—

$$(101.)\quad \int_{-1}^{+1}\frac{\epsilon^{a+bx+cx^2+ex^3}}{\sqrt{1-x^2}}dx=\pi(A_0^2+A_1^2+A_2^2+\ldots)$$

where A_0, A_1, A_2 are the coefficients of expansion of

$$\epsilon^{\frac{a}{2}+\frac{c}{4}+\left(\frac{b}{2}+\frac{3e}{8}\right)x+\frac{c}{4}x^2+\frac{e}{8}x^3}.$$

Similarly we may find

$$(102.)\quad \int_{-1}^{+1}\frac{\epsilon^{a+bx+cx^2+\cdots+ex^2}}{\sqrt{1-x^2}}\,dx.$$

$$(103.)\quad \int_{0}^{\pi}\frac{\epsilon^{\cos\theta}d\theta}{1-2\alpha\cos\theta+\alpha^2}$$

$$=\frac{\pi}{1-\alpha^2}\left\{1+\frac{1}{2^2}+\frac{1}{1^2\,.\,2^2}\cdot\frac{1}{2^4}+\frac{1}{1^2\,.\,2^2\,.\,3^2}+\frac{1}{2^6}+\ldots\right\}$$

$$+\frac{2\alpha\pi}{1-\alpha^2}\left\{\frac{1}{2}+\frac{1}{1^2\,.\,2}\cdot\frac{1}{2^3}+\frac{1}{1^2\,.\,2^2\,.\,3}\cdot\frac{1}{2^5}+\ldots\right\}$$

$$+\frac{2\alpha^2\pi}{1-\alpha^2}\left\{\frac{1}{1\,.\,2}\cdot\frac{1}{2^2}+\frac{1}{1^2\,.\,2\,.\,3}\cdot\frac{1}{2^4}+\frac{1}{1^2\,.\,2^2\,.\,3\,.\,4}+\ldots\right\}$$

$$+\ldots$$

$$=\frac{2\pi}{1-\alpha^2}\left\{\epsilon^{\frac{\alpha}{2}}+\frac{1}{2\alpha}(\epsilon^{\frac{\alpha}{2}}-1)+\frac{1}{1\,.\,2\,.\,2^2\,.\,\alpha^2}(\epsilon^{\frac{\alpha}{2}}-\frac{\alpha}{2}-1)+\ldots\right\}$$

$$-\frac{\pi}{1-\alpha^2}\left\{1+\frac{1}{2^2}+\frac{1}{1^2\,.\,2^2}\cdot\frac{1}{2^4}+\frac{1}{1^2\,.\,2^2\,.\,3^2}\cdot\frac{1}{2^6}+\ldots\right\}.$$

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(104.) $$\int_0^\pi \cos\cos\theta d\theta = \int_0^\pi \cos\sin\theta d\theta$$

$$= \pi\left\{1 - \frac{1}{2^2} + \frac{1}{1^2 . 2^2}\cdot\frac{1}{2^4} - \frac{1}{1^2 . 2^2 . 3}\cdot\frac{1}{2^6} + \ldots\right\}.$$

(105.) $$\int_0^\pi \epsilon^{\cos\theta}\, d\theta = \int_0^\pi \epsilon^{\sin\theta}\, d\theta$$

$$= \pi\left\{1 + \frac{1}{2^2} + \frac{1}{1^2 . 2^2}\cdot\frac{1}{2^4} + \frac{1}{1^2 . 2^2 . 3^2}\cdot\frac{1}{2^6} + \ldots\right\}.$$

In the same manner we may find

(106.) $$\int_0^\pi \frac{\epsilon^{\cos\theta}\, d\theta}{(1 + 2\alpha\cos\theta + \alpha^2)^{\frac{1}{2}}}.$$

(107.) $$\int_0^\pi \cos r\theta d\theta \sqrt{a + b\cos\theta + c\cos^2\theta + c\cos^3\theta}$$

$$= \pi(\mathrm{A}_0\mathrm{A}_r + \mathrm{A}_1\mathrm{A}_{r+1} + \mathrm{A}_2\mathrm{A}_{r+2} + \ldots)$$

when A_0, A_1, A_2 . . . are the coefficients of expansion of

$$\sqrt{p + qx + rx^2 + sx^3},$$

p, q, r, s being determined by the equations

$$p^2 + q^2 + r^2 + s^2 = a + \frac{c}{2}, \qquad 2pq + 2qr + 2rs = b + \frac{3e}{4},$$

$$2pr + 2qs = \frac{c}{2}, \qquad 2ps = \frac{e}{4}.$$

We are able of course to deduce

(108.) $$\int_0^\pi d\theta \cos^r\theta \sqrt{(a + b\cos\theta + c\cos^2\theta + e\cos^3\theta)},$$

and similarly the more general integrals

(109.) $$\int_0^\pi d\theta \cos^r\theta \sqrt{(a + b\cos\theta + c\cos^2\theta + \ldots c\cos^m\theta)},$$

and also

(110.) $$\int_0^\pi d\theta \cos^r\theta^m \sqrt[m]{(a + b\cos\theta + c\cos^2\theta + \ldots + c\cos^m\theta)}.$$

Since the following integral is known :—

$$\int_0^\omega \frac{dx \sin rx}{\epsilon^{\pi x} - \epsilon^{-\pi x}} = \frac{1}{4}\,\frac{\epsilon^r - 1}{\epsilon^r + 1},$$

we obtain at once

$$\int_0^\omega \frac{\alpha^r \epsilon^r \sin rx dx}{\epsilon^{\pi x} - \epsilon^{-\pi x}} + \int_0^\omega \frac{\alpha^r \sin rx dx}{\epsilon^{\pi x} - \epsilon^{-\pi x}} = \frac{\alpha^r \epsilon^r}{4} - \frac{\alpha^r}{4}.$$

Hence, if we put

$$\phi(\rho) = \int_0^\omega \frac{dx\, \rho \sin x}{(1-2\rho\cos x+\rho^2)(\epsilon^{\pi x} - \epsilon^{-\pi x})}$$

we have
$$\phi(\rho) + \phi\left(\frac{\rho}{\epsilon}\right) = \frac{1}{4}\frac{\rho}{1-\rho} - \frac{1}{4}\cdot\frac{\frac{\rho}{\epsilon}}{1-\frac{\rho}{\epsilon}},$$

so also
$$\phi\left(\frac{\rho}{\epsilon}\right) + \phi\left(\frac{\rho}{\epsilon^2}\right) = \frac{1}{4}\frac{\frac{\rho}{\epsilon}}{1-\frac{\rho}{\epsilon}} - \frac{1}{4}\frac{\frac{\rho}{\epsilon}}{1-\frac{\rho}{\epsilon}},$$

and continuing the process

$$(111.)\quad \phi(\rho) = \frac{1}{4}\cdot\frac{\rho}{1-\rho} - \frac{1}{2}\frac{\frac{\rho}{\epsilon}}{1-\frac{\rho}{\epsilon}} + \frac{1}{2}\frac{\frac{\rho}{\epsilon^2}}{1-\frac{\rho}{\epsilon^2}} - \alpha.$$

In like manner we obtain

$$(112.)\quad \int_0^\omega \frac{dx \epsilon^{\rho\cos x}\sin(\rho\sin x)}{\epsilon^{\pi x} - \epsilon^{-\pi x}} = \frac{1}{4}\epsilon^\rho - \frac{1}{2}\epsilon^{\frac{\rho}{\epsilon}} + \frac{1}{2}\epsilon^{\frac{\rho}{\epsilon^2}} - \frac{1}{2}\epsilon^{\frac{\rho}{\epsilon^3}} + \ldots$$

In a similar way also we may find the values of

$$(113.)\quad \int_0^\omega \frac{dx}{(1-2\alpha\cos x+\alpha^2)(\epsilon^{\pi x}+\epsilon^{-\pi x})},\ \text{and}$$

$$(114.)\quad \int_0^\omega \frac{dx \epsilon^{\alpha\cos x}\cos(\alpha\sin x)}{\epsilon^{\pi x}+\epsilon^{-\pi x}}.$$

Let us now return for a moment to the equations

$$p^2+q^2+r^2+s^2=\alpha. \qquad 2pq+2qr+2rs=\beta.$$

$$2pr+2qs=\gamma. \qquad 2ps=\delta.$$

Then $\alpha+\beta+\gamma+\delta$ gives us

$$p+q+r+s=\sqrt{(\alpha+\beta+\gamma+\delta)},$$

$\alpha-\beta+\gamma-\delta$ gives us

$$p-q+r-s=\sqrt{(\alpha-\beta+\gamma-\delta)},$$

hence we obtain at once $p+r$, and $q+s$, which will render the solution of the equations easy, at all events, in all possible cases.

I would moreover remark that the validity of the process depends on $\surd(p+qx+rx^2+sx^3)$ expanding in a converging series, so that the method of evaluation here, α of course in (109), (110), depends on certain conditions, to which the constants in the integrals must be subject.

II. "On certain Definite Integrals." No. 7. By W. H. L. RUSSELL, F.R.S. Received January 6, 1880.

By a development of the methods indicated in the former papers we obtain the following integrals:—

(115.) $$\int_0^{\frac{\pi}{2}} d\theta\,.\,\frac{\cos^n\theta\cos(n-2)\theta+\alpha\cos^{n+1}\theta\cos(n-3)\theta}{\sin^2\theta+(1+\alpha)^2\cos^2\theta}$$
$$=\frac{\pi}{2^n}\left\{\frac{n}{\alpha+2}-\frac{\alpha}{(\alpha+2)^2}\right\}.$$

(116.) $$\int_0^{\frac{\pi}{2}} d\theta\cos^2\theta\,\epsilon^{2x\cos^2\theta}\cos(x\sin 2\theta)=\frac{\pi}{8}(x+2)\epsilon^x.$$

(117.) $$\int_0^{\frac{\pi}{2}} d\theta\,\frac{\cos^n\theta\cos(n-2)\theta+\alpha^3\cos^{n+3}\theta\cos(n-5)\theta}{1+2\alpha^3\cos^3\theta\cos 3\theta+\alpha^6\cos^6\theta}$$
$$=\frac{\pi}{2^{n-2}}\left\{\frac{n}{\alpha^3+8}-\frac{3\alpha^3}{(a^3+8)^2}\right\}$$

(118.) $$\int_0^{\frac{\pi}{2}} d\theta\cos^2\theta\,\epsilon^{\alpha\cos^3\theta\cos 3\theta}\cos(\alpha\cos^3\theta\sin 3\theta)=\frac{\pi}{8}\left(2+\frac{3\alpha}{8}\right)\epsilon^{\frac{\alpha}{8}}.$$

(119.) $$\int_0^{\frac{\pi}{2}}\frac{\cos^\mu\theta\cos\mu\theta+\alpha\cos^{\mu+1}\theta\cos(\mu-1)\theta}{\sin^2\theta+(\alpha+1)^2\cos^2\theta}d\theta=\frac{\pi}{2^\mu}\frac{1}{\alpha+2}.$$

(120.) $$\int_0^{\frac{\pi}{2}} d\theta\frac{\cos^\mu\theta\cos\mu\theta+(\alpha+\beta)\cos^{\mu+\theta}\cos(\mu-1)\theta+\alpha\beta\cos^{\mu+1}\theta\cos(\mu-2)\theta}{(\sin^2\theta+(\alpha+1)^2\cos^2\theta)(\sin^2\theta+(\beta+1)^2\cos^2\theta}$$
$$=\frac{\pi}{2^{\mu-1}(\alpha+2)(\beta+2)}.$$

This integral may be written

(121.) $$\int_0^{\frac{\pi}{2}} d\theta\frac{\cos^\mu\theta\cos\mu\theta+p\cos^{\mu+1}\theta\cos(\mu-1)\theta+q\cos^{\mu+2}\theta\cos(\mu-2)\theta}{\sin^4\theta+(p^2+2p-2q+2)\sin^2\theta\cos^2\theta+(p+q+1)^2\cos^4\theta}$$
$$=\frac{\pi}{2^{\mu-1}(q+2p+4)}$$